100 POINTS | MATH 5B - TEST 4 | Spring 2024

(11.8-11.11 , 8.1 10.1 10.3)

NAME: ______________________

One page of formulas allowed. Show all work clearly on test paper. No credit will be given for solutions if work is not shown. Unless otherwise specified, the answer to series questions should be given using sigma notation. Unless otherwise stated, you do not need to find the radius of convergence.

(1) Find the interval of convergence for each of the following. Explain. (8 points each)

(a) $\sum_{n=0}^{\infty} \frac{2^n}{n!} x^n$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{2^n x^n} \right| = \lim_{n\to\infty} \frac{2}{n+1}|x| = 0$$

for all x.

$$\boxed{(-\infty, \infty)}$$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \sqrt{n}} (x-1)^n$

Ratio Test

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (x-1)^{n+1}}{2^{n+1}\sqrt{n+1}} \cdot \frac{2^n \sqrt{n}}{(-1)^n (x-1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{|x-1|}{2} \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{|x-1|}{2} \underbrace{\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}}}_{1} = \frac{|x-1|}{2}$$

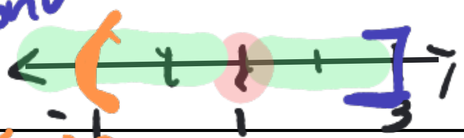
Series converges when $\frac{|x-1|}{2} < 1 \Rightarrow |x-1| < 2$

When $\frac{|x-1|}{2} = 1$, Ratio Test is inconclusive and we must check endpoints separately

$\Rightarrow -2 < x-1 < 2$

$-1 < x < 3$

$x=3$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \sqrt{n}} (3-1)^n = \sum \frac{(-1)^n}{\sqrt{n}}$ conv



$x=-1$: $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n \sqrt{n}} (-1-1)^n = \sum \frac{(-1)^n (-2)^n}{2^n \sqrt{n}} = \sum \frac{1}{\sqrt{n}}$ div.

$$\boxed{\text{Ans}: (-1, 3]}$$

(c) $\sum_{n=0}^{\infty} (3n)! (x+2)^n$

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(3(n+1))! (x+2)^{n+1}}{(3n)! (x+2)^n} \right|$$

$$= \lim_{n\to\infty} |(3n+3)(3n+2)(3n+1)(x+2)| = \infty$$ for all x but $x = -2$.

So converges $\boxed{\{-2\}}$

(2) Find the Maclaurin series for $\dfrac{1}{1+4x}$ using any method and state the radius of convergence.

(8 points)

directly or use geometric series

easiest

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad |x|<1$$

sub. $(-4x)$ for x

$$\frac{1}{1+4x} = \sum_{n=0}^{\infty} (-4x)^n \qquad |-4x|<1$$

$$\frac{1}{1+4x} = \boxed{\sum_{n=0}^{\infty} (-4)^n x^n} \qquad |x| < \frac{1}{4}$$

$$\boxed{R = \frac{1}{4}}$$

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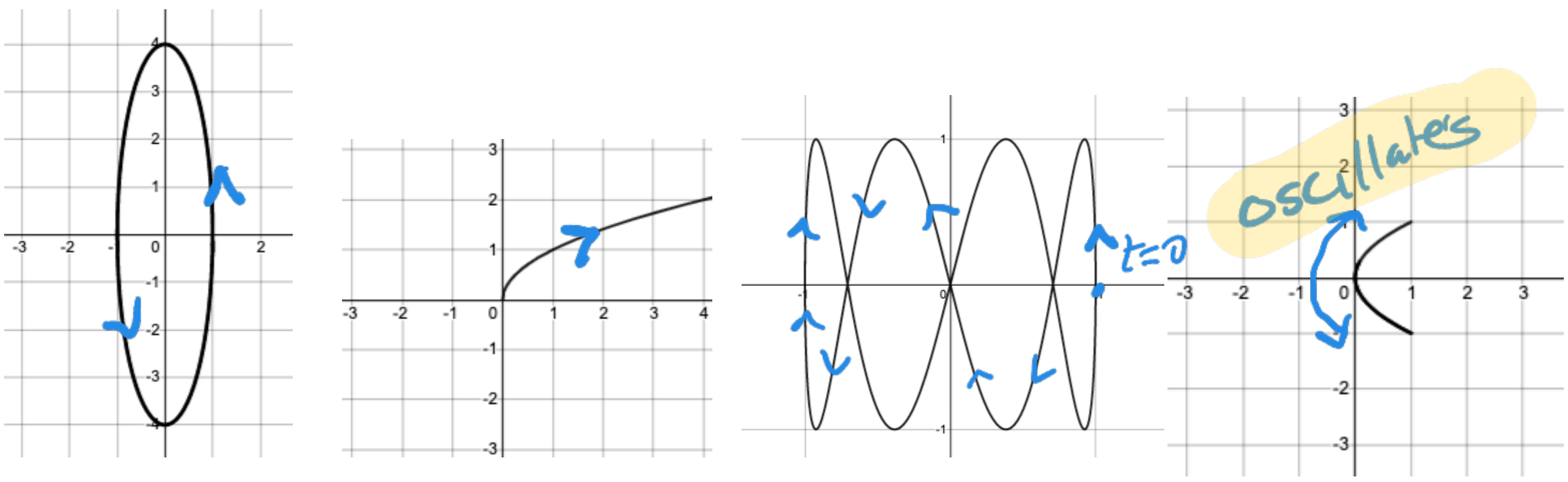
(3) Match the parameterizations (a)-(d) with their plots shown **and draw an arrow indicating direction of increasing t.** (8 points)

(a) $\begin{cases} x = \cos t \\ y = \sin 4t \end{cases}$ C (b) $\begin{cases} x = t \\ y = \sqrt{t} \end{cases}$ B (c) $\begin{cases} x = \cos t \\ y = 4\sin t \end{cases}$ A (d) $\begin{cases} x = \cos^2 t \\ y = \cos t \end{cases}$ D



A B C D

(4) Find the Taylor series for $f(x)=\dfrac{4}{x^3}$ centered at a = 3 . Express in Summation Notation. No need to find radius of convergence.

(10 points)

a) Direct Approach, using the defini

	at x	at x=a=0 3
f	$4x^{-3}$	
f '	$-4\ \ 3x^{-4}$	
f ''	$4\ \ 4\cdot 3x^{-5}$	
f '''	$-4\ \ 5\cdot 4\cdot 3x^{-6}$	
f ⁿ	f ⁿ(x)= $\dfrac{(-1)^n 4\,(n+2)!}{2x^{n+3}}$	f ⁿ(0)=

notice, this is not quite a factorial. It needs a 2.

$$\frac{5\cdot 4\cdot 3\cdot 2}{2}=\frac{5!}{2}$$

$$f^{(n)}(x)=\frac{(-1)^n 4\,(n+2)!}{x^{n+3}\cdot 2}$$

$$f^{(n)}(3)=\frac{(-1)^n 4\,(n+2)!}{3^{n+3}\cdot 2}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n 4\,(n+2)!}{3^{n+3}\cdot 2\ n!}(x-3)^n$$

Be sure to put the 3 in for x, otherwise you'd have an x here, which isn't a power series (not polynomial form)

$$=\boxed{\sum_{n=0}^{\infty}\frac{(-1)^n 2\,(n+2)(n+1)}{3^{n+3}}(x-3)^n}$$

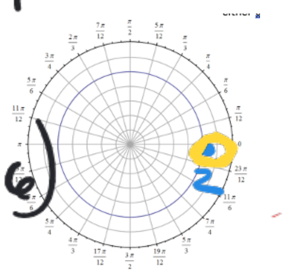
(5) (12 points) 4

(a) Convert from polar to rectangular coordinates $(-2, \pi)$: $(2, 0)$

$x = r\cos\theta = -2\cos\pi$

$y = r\sin\theta = -2\sin\pi$

(can check if reasonable)

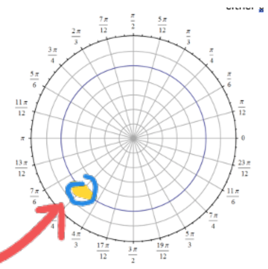


(b) Convert from rectangular to polar coordinates $(-3, -3)$: $\left(3\sqrt{2}, \frac{5\pi}{4}\right)$

$r^2 = x^2 + y^2 = 18 \quad r = 3\sqrt{2}$

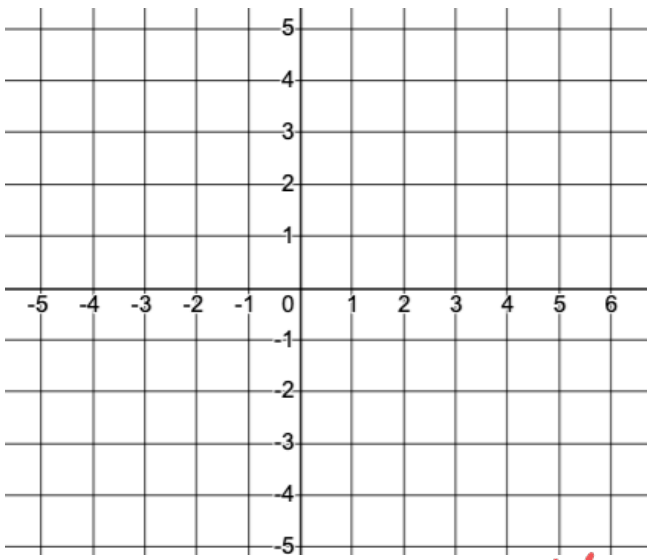
$\tan\theta = \frac{-3}{-3} = 1$, Q in Q3 $\theta = \frac{5\pi}{4}$

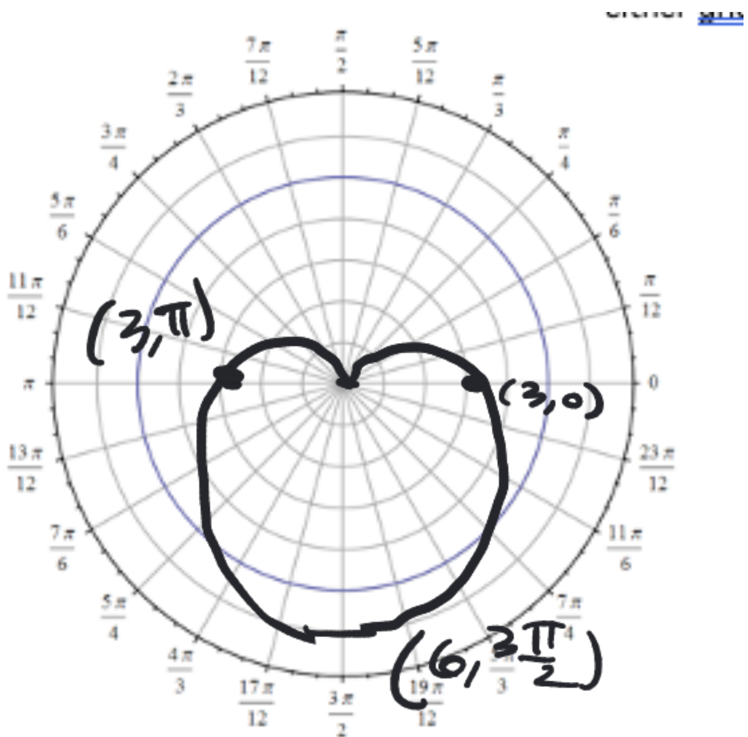
* point in Q3



(c) Graph the polar function: $r = 3 - 3\sin\theta$ (You can use either grid)

cardioid





Note - very round, not ♡

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(6) Find a_5 , the fifth term of the series (no need to simplify).

$$\sum_{n=1}^{\infty} \frac{n!}{1 \bullet 5 \bullet 9 \bullet \cdots \bullet (4n-3)} x^n$$

$n = 5 \Rightarrow 4n - 3 = 20 - 3 = 17$ (3 points)

$$\boxed{a_5 = \frac{5!}{1 \cdot 5 \cdot 9 \cdot 13 \cdot 17} x^5}$$

(7) (21 points)

(a) Use the Maclaurin series for e^x to find the Maclaurin Series for $f(x) = x^2 e^{-x^2}$

Easiest to use known series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} = \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

subst. $-x^2$ for x

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

mult x^2

$$x^2 e^{-x^2} = x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \frac{x^{10}}{4!} + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}}$$

(b) Use series to compute $\int_0^{1/3} x^2 e^{-x^2}\, dx$ with $|\text{error}| < 0.001$

Show how you determined how many terms were necessary

$$\int_0^{1/3} x^2 e^{-x^2} dx = \int_0^{1/3} \left(x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \frac{x^{10}}{4!} + \right) dx = \int_0^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!} dx$$

$$= \left[\frac{1}{3}x^3 - \frac{x^5}{5} + \frac{x^7}{14} - \frac{x^9}{54} + \frac{x^{11}}{264} - \cdots \right]_0^{1/3}$$

$$= \frac{1}{81} - \frac{1}{1215} + \cdots$$

first term < .001 *

$$\approx \boxed{\frac{1}{81} = .0123}$$

*Note: You include all terms that precede the first term < .001 so you only need one term here.

(8) Use an integral to find the length of the curve $f(x) = (x-1)^{3/2}$; ~~$0 \le x \le 2$~~
$1 \le x \le 5$

$$f'(x) = \frac{3}{2}(x-1)^{1/2}$$

$$L = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_1^5 \sqrt{1+\left(\frac{3}{2}(x-1)^{1/2}\right)^2}\,dx$$

$$= \int_1^5 \sqrt{1+\frac{9}{4}(x-1)}\,dx$$

$$= \int_1^5 \sqrt{\frac{4+9(x-1)}{4}}\,dx$$

$$= \frac{1}{2}\int_1^5 \sqrt{9x-5}\,dx$$

let $u = 9x-5$
$du = 9\,dx$

$$= \frac{1}{18}\int_4^{40} u^{1/2}\,du$$

$$= \frac{1}{18}\cdot\frac{2}{3}u^{3/2}\Big]_4^{40}$$

$$= \frac{1}{27}\left(40^{3/2}-4^{3/2}\right)$$

$$= \frac{1}{27}\cdot 4^{3/2}\left(10^{3/2}-1\right)$$

$$= \boxed{\frac{8}{27}\left(10^{3/2}-1\right)}$$